A Modification of Contour-Improved Perturbation Theory

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Abstract

We present a modification of the standard method of evaluating the semihadronic tau decay width. The method is based on a derivative expansion for the Adler function rather than the standard series in powers of the strong coupling. The extracted QCD coupling at the tau mass scale is by 2% lower than the "contour improved" value. We find $\alpha_s(M_Z^2) = 0.1211 \pm 0.0010$.

1. Introduction

Within[5] perturbative QCD, physical quantities are expressed as (truncated) standard series in powers of $a(Q^2) \equiv \alpha_s(Q^2)/\pi$. We reorganize such power series into series of tilde-couplings $\tilde{a}_n(Q^2)$, where $\tilde{a}_n(Q^2)$ are logarithmic derivatives of $a(Q^2)$ normalized so that $\tilde{a}_n = a^n + O(a^{n+1})$. The requirement for the new series in terms of the tilde couplings is that it should reproduce the old one order by order. Considering the two infinite series should be, at least formally, also equivalent. After truncation, the difference between the series is of the order of the next (unknown) term. For a small enough coupling a this difference is negligible when comparing against experiments. However, as we shall see, the difference is relevant in the case of the semihadronic tau decay width, where the coupling is not small and the convergence of the original series is poor.

The ratio of semihadronic to leptonic tau decay widths

$$R_\tau = \frac{\Gamma(\tau \to \text{had } \nu_\tau (\gamma))}{\Gamma(\tau \to e^- \bar{\nu}_e \mu_\tau (\gamma))}, \qquad (1)$$

can be separated experimentally, identifying final states with net strangeness, and final states without net strangeness. Within the latter, the vector (V) from the axial vector (A) channels can be also distinguished: $R_\tau = R_\tau^S + R_\tau^V + R_\tau^A$.

We shall extract the strong coupling a essentially from the vector plus axial channel

$$R_\tau^{V+A} \equiv R_\tau^V + R_\tau^A = 3.479 \pm 0.011. \qquad (2)$$

where we quoted its experimental value [2]. The various contributions to the last quantity can be identified using

$$R_\tau^{V+A} = 3|V_{ud}|^2 S_{ew}(1 + \delta_0 + \delta'_{ew} + \delta_2 + \delta_{NP}), \qquad (3)$$

where the perturbative QCD correction, considered in this article, is given by δ_0. $S_{ew} = 1.0198 \pm 0.0006$ [3] and $\delta'_{ew} = 0.001 \pm 0.001$ [4] are electroweak corrections, $\delta_2 = (-4.3 \pm 2.0) \times 10^{-4}$ are light quark masses effects, $\delta_{NP} = (-5.9 \pm 1.4) \times 10^{-3}$ [2] are nonperturbative contributions, and $V_{ud} = 0.97418 \pm 0.00027$ [5]. From these values and the value in eq. (2) we obtain the experimental prediction for δ_0

$$\delta_0 = 0.204 \pm 0.004 . \qquad (4)$$

From this value, α_s can be extracted using only perturbative QCD.

The perturbative QCD contribution can be expressed in terms if a contour integral

$$\delta_0 = \frac{1}{2\pi i} \oint_{|x|=1} \frac{dx}{x} (1-x)^3 (1+x) \hat{D}(-xM_\tau^2), \qquad (5)$$

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where $\hat{D}(Q^2)$ is the canonically normalized Adler function. Its massless perturbative QCD (leading twist) contribution is given by

$$\hat{D}(Q^2) = \sum_{n=1}^{4} a^n(\mu^2) \sum_{m=0}^{n-1} c_{n,m} \log^m(Q^2/\mu^2), \quad (6)$$

where μ is the renormalization scale, and $c_{n,m}$ the expansion coefficients. The coefficients $c_{n,0}$ are mutually independent. The use of the renormalization group (RG) equation gives us the coefficients $c_{n,m}$ with $m \geq 1$ as linear combinations of the coefficients $c_{n',0}$ with $n' < n$. These relations involve also coefficients of the perturbative β-function. The Adler function is a space-like observable. Its corresponding expression at $\mu^2 = Q^2$ ("RG-improved" expression) is

$$\hat{D}^{RG}(Q^2) = \sum_{n=1}^{4} c_{n,0} \, a^n(Q^2). \quad (7)$$

The evaluation of δ_0 by means of eqs. (5) and (7) is known as Contour-Improved Perturbation Theory (CIPT). This is the standard method for the evaluation of R_τ. We shall compare it against its proposed modification in the next section.

2. Modified CIPT

In the modified CIPT approach, as in the (standard) CIPT approach, the semihadronic tau decay ratio is evaluated in eq. (5) using for the Adler function $\hat{D}$ a RG-improved expression. However, in the modified approach we use, instead of the standard series in powers of $a(Q^2)$ given in eq. (7), a nonpower expansion for the Adler function in terms of the new *couplings* $\tilde{a}_n(Q^2)$. Truncated at the last known term the new expansion is given by

$$\tilde{D}(Q^2) = \sum_{n=1}^{4} \tilde{c}_n \, \tilde{a}_n(Q^2), \quad (8)$$

with the tilde couplings defined as

$$\tilde{a}_{m+1} \equiv \frac{(-1)^m}{\beta_0^m \, m!} \frac{d^m a}{d(\log Q^2)^m}. \quad (9)$$

The derivatives are evaluated perturbatively using the β-function

$$\frac{\partial a}{\partial \log \mu^2} = \beta(a) = -(\beta_0 \, a^2 + \beta_1 \, a^3 + \beta_2 \, a^4 + \beta_3 \, a^5). \quad (10)$$

The new coefficients $\tilde{c}_n$ are obtained from the coefficients $c_{m,0}$ with $m \leq n$. The tilde couplings are normalized such that $\tilde{a}_n = a^n + O(a^{n+1})$. Note that in the new expansion $\tilde{a}_1 = a$, $\tilde{a}_2 = -\beta(a)/\beta_0$, $\tilde{a}_3 = \beta(a)\beta'(a)/(2\beta_0^2)$, etc. The beta function and its derivatives play in (8) a more central role than in eq. (7). The ratio $\tilde{a}_n/a^n$ grows with n:

$$\frac{\tilde{a}_n}{a^n} = 1 + r_n + O(a^2), \quad (11)$$

where

$$r_{n+1} = r_n(n+1)/n + \beta_1/\beta_0. \quad (12)$$

For low values of a and n we have $\tilde{a}_n/a^n$ near 1. The tilde and the standard couplings have similar analyticity properties. If we restrict ourselves to real Q^2, there are poles and cuts in the infrared region.

The series (7) and (8) are formally equal if infinite number of terms in both expressions is considered. However, these series are believed to be asymptotic. If we had the complete perturbative series we would truncate them at their respective (in absolute value) smallest terms in order to give a meaning to the sums. The difference between the sum and the exact value of the original quantity is expected to be smaller than the last considered term. Therefore, the rearrangement or reshuffling of eq. (7) in eq. (8) is relevant and it leads in general to different values of the truncated series for the Adler function. Further, with the coefficients known at present, the asymptotic behavior of the Adler function is possibly not reached and we have to truncate the series before, obtaining also different values in CIPT and modified CIPT.

We will evaluate R_τ in the modified CIPT, and will compare the results with those of the (standard) CIPT approach. The Adler function coefficients $c_{n,0}$, in $\overline{MS}$ scheme and for three active flavors, are

$$\begin{array}{ll} c_{1,0} = 1, & \tilde{c}_1 = 1, \\ c_{2,0} = 1.6398, & \tilde{c}_2 = 1.6398, \\ c_{3,0} = 6.3710, & \tilde{c}_3 = 3.4558, \\ c_{4,0} = 49.076, & \tilde{c}_4 = 26.385. \end{array} \quad (13)$$

The first two coefficients are equal by construction and the third and fourth tilde coefficients are about half the standard ones.

We extract a from the experimental value $\delta_0 = 0.204$ in $\overline{MS}$ scheme using the expression (5), eq. (7) for CIPT (CI), and eq. (8) for modified CIPT ($\overline{CI}$), obtaining

δ_0	CI	$\overline{CI}$	
1	0.1513	0.1484	
2	0.0308	0.0372	
3	0.0128	0.0104	(14)
4	0.0090	0.0078	
$\sum_{n=1}^{4}$	0.2038	0.2039	
Extracted a	$0.347/\pi$	$0.341/\pi$	

In CIPT the experimental value is reproduced if $a^{\mathrm{CI}}(M_\tau^2) = 0.347/\pi$, while in modified CIPT a lower value is required, $a^{\mathrm{mCI}}(M_\tau^2) = 0.341/\pi$ (a different value of a is used in each column). We conclude that independent of the precise value of δ_0 in modified CIPT we get a value of the strong coupling at the τ mass scale by about 2% lower than in CIPT. In addition, we observe in modified CIPT compared to CIPT a smaller last term of the series.

Further, we investigate the renormalization scale dependence of the evaluated expressions δ_0, in $\overline{\mathrm{MS}}$ scheme, evaluated in the CIPT and modified CIPT approach. The (squared) renormalization scale is chosen to be $\mu^2 = \xi M_\tau^2$ with $\xi = 0.7$, 1, and 2, and we use $a(M_\tau^2) = 0.34/\pi$:

ξ	$a(\xi M_\tau^2)$	δ_0, CI	$\delta_0, \overline{CI}$	
0.7	$0.3831/\pi$	0.2009	0.2020	
1	$0.3400/\pi$	0.1984	0.2031	(15)
2	$0.2812/\pi$	0.1907	0.1991	

If we take as a measure of the scale dependence of δ_0 its range of variation when ξ varies between 0.7 and 2, we obtain 0.0102 in the CIPT and 0.0040 in the modified CIPT approach. If we extract α_s from the experimental value of δ_0, these uncertainties translate to $\Delta\alpha(M_\tau^2) = 0.013$ in the CIPT and 0.005 in the modified CIPT approach. If we use this criterion, the renormalization scale dependence is by more than factor two weaker in the modified CIPT than in the standard CIPT. This is a very attractive feature of modified CIPT.

In a similar analysis we conclude that the renormalization scheme dependence (i.e. different choices for the beta function coefficients, β_n with $n \geq 2$) is similar in both methods.

The modified CIPT possesses some attractive properties, among them a lower renormalization scheme dependence as seen above. From the point of view of the standard power series of the Adler function, the modified CIPT performs the sum (7) up to $n = 8$ considering nonzero coefficients $c_{n,0}$ for $n = 5$ to 8. For example, when expanding the truncated expression (8) in powers of a, we obtain $c_{5,0} = 300.4$. The question can be asked how this value compares with the exact one. For this we

need to calculate the corresponding Feynman diagrams, a task not expected to be done in the near future. However, as a test of the method we can compare its prediction for the known coefficients $c_{3,0}$ and $c_{4,0}$. From $c_{1,0}$ and $c_{2,0}$ the estimate for $c_{3,0}$ is 2.92, and from $c_{1,0}$, $c_{2,0}$ and $c_{3,0}$ the estimate for $c_{4,0}$ is 22.7. When we compare these with the exact values, cf. eq. (13), we conclude that in both cases the modified CIPT includes a significant part of the next term of the power series. At least in these two cases, the modified CIPT is an improvement also from this point of view. Both estimates (for $c_{3,0}$ and $c_{4,0}$) are lower than the exact value by about a factor of 2.2. If we use the same correction factor we obtain the prediction $c_{5,0} = 300.4 \times 2.2 = 661$.

3. Extraction of α_s

The uncertainty in the value of the pseudo-observable quantity δ_0 given in eq. (4) results in the experimental uncertainty in the extraction of α_s, $\Delta\alpha^{\mathrm{exp}} = \pm 0.005$. The main contribution here is due to the experimental value of R_τ^{V+A}, given in eq. (2). The nonperturbative contribution δ_{NP} also contributes to the experimental uncertainty of α_s. Its value and its associated uncertainty are small, cf. Ref. [2]. The uncertainty in δ_{NP} has almost no relevance for the uncertainty of δ_0. On the other hand, the effect of δ_{NP} on the central value of δ_0 is 0.006, i.e. 1.5 times δ_0's experimental uncertainty.

Variations of the renormalization scale and scheme, within the modified CIPT, lead to the uncertainties $\Delta\alpha^{\mathrm{scl}} = 0.005$ and $\Delta\alpha^{\mathrm{sch}} = 0.004$. Adding them in quadrature gives $\Delta\alpha^{\mathrm{theo}} = 0.006$. As a consequence, the extracted value of the strong coupling constant at the m_τ scale in the modified CIPT is

$$
\begin{aligned}
\alpha_s^{\mathrm{mCI}}(M_\tau^2) &= 0.341 \pm 0.005^{\mathrm{exp}} \pm 0.006^{\mathrm{theo}}, \quad (16) \\
&= 0.341 \pm 0.008.
\end{aligned}
$$

The corresponding value in the (standard) CIPT approach is

$$
\begin{aligned}
\alpha_s^{\mathrm{CI}}(M_\tau^2) &= 0.347 \pm 0.005^{\mathrm{exp}} \pm 0.014^{\mathrm{theo}}, \quad (17) \\
&= 0.347 \pm 0.015.
\end{aligned}
$$

An alternative way to estimate the theoretical uncertainty for α_s would be to regard this uncertainty as coming uniquely from the way we use the RG: the difference between the extracted central values using the CIPT and the modified CIPT approach. It's interesting that this would lead to the same theoretical uncertainty ± 0.006

obtained above within the modified CIPT. We are not al-
lowed to combine the two uncertainties since this would
represent a double counting. This is so because, by
definition, the difference between the truncated CIPT
and the truncated modified CIPT approach is given by
higher order contributions.

When we evolve by RG from the scale M_τ to the scale
M_Z, we obtain in the modified CIPT

$$\alpha_s^{\mathrm{mCI}}(M_Z^2) = 0.1211 \pm 0.0010 . \tag{18}$$

Here, the experimental, theoretical and the RG evolu-
tion and matching uncertainties are included.

4. Conclusions

We presented a modification of the Contour-
Improved Perturbation Theory (CITP), i.e., modifica-
tion of the standard method for the evaluation of the
semihadronic tau decay width ratio R_τ. The truncated
result (truncated at $\sim \alpha_s^4$) is sensitive to higher order
terms, principally because the momenta involved are
low $\sim M_\tau$ (≈ 1.8 GeV). Important uncertainties in the
evaluation of R_τ come from the way we use the renor-
malization group in calculating R_τ from the Adler func-
tion. For example, it is well known that the CIPT and
the fixed-order perturbation theory (FOPT) approach
give significantly different results. Both in the CIPT and
in the proposed modified CIPT approaches the Adler
function $D(Q^2)$ is evaluated using a "running" renor-
malization scale $\mu^2 = Q^2 = M_\tau^2 \exp(i\theta)$ in the con-
tour integral, and not a fixed one $\mu^2 = M_\tau^2$ as in FOPT.
While the (standard) CIPT uses for the Adler function
$D(Q^2)$ along the contour the usual (truncated) power se-
ries $a + c_1 a^2 + c_2 a^3 + \dots$ (where $a \equiv a(Q^2) \equiv \alpha_s(Q^2)/\pi$),
the modified CIPT uses a (truncated) nonpower series
$a + \tilde{c}_1 \tilde{a}_2 + \tilde{c}_2 \tilde{a}_3 + \dots$. Here, the new tilde couplings $\tilde{a}_{n+1}$
are proportional to the n'th logarithmic derivative of the
coupling $a \equiv a(Q^2)$, and $\tilde{c}_n$ are the new expansion co-
efficients. This expansion, in derivatives of α_s, was first
introduced in Ref. [6] in the context of analytic QCD
models.

The modified CIPT approach has several advantages
in comparison to the standard CIPT. Among them, the
dependence of the truncated result for the semihadronic
tau decay ratio on the variation of the renormalization
scale $\mu^2 = \xi^2 M_\tau^2 \exp(i\theta)$ in the contour integral is by
about factor of two weaker than in the standard CIPT
approach. Further, the last term of the expansion in the
aforementioned decay ratio is reduced by about 10%.
The renormalization scheme dependence remains ap-
proximately equal.

The extracted values of $\alpha_s(M_\tau^2)$ and $\alpha_s(M_Z^2)$ from the
nonstrange $R_\tau(V + A)$ ratio are in the modified CIPT
(mCI) approach by 1.8% and 0.5% lower than in the
CIPT approach, respectively. Our result is $\alpha_s^{\mathrm{mCI}}(M_\tau^2) =$
0.341 ± 0.008 and $\alpha_s^{\mathrm{mCI}}(M_Z^2) = 0.1211 \pm 0.0010$.

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